Chapter 3

Second Order Linear Differential Equations

Sec 3.1: Homogeneous equation

A linear second order differential equations is written as

\[ a(x)y'' + b(x)y' + c(x)y = d(x). \]

When \( d(x) = 0 \), the equation is called **homogeneous**, otherwise it is called **nonhomogeneous**.

\[
\begin{align*}
\text{(NH)} & \quad a(x)y'' + b(x)y' + c(x)y = d(x), \\
\text{(H)} & \quad a(x)y'' + b(x)y' + c(x)y = 0.
\end{align*}
\]

For the study of these equations sometimes we will consider the **standard forms** given by

\[
\begin{align*}
y'' + p(x)y' + q(x)y & = g(x) \\
y'' + p(x)y' + q(x)y & = 0
\end{align*}
\]

**Linear differential operators (ways to write the linear 2\textsuperscript{nd} ODE):**

In mathematics a function that ``transforms a function into a different function'' is called an **operator**.

(1) **L-Operator**

Let \( y'' + p(x)y' + q(x)y = g(x) \)

be a second order linear differential equation. Then we call the operator

\[ L(y) = y'' + p(x)y' + q(x)y \]

the **corresponding linear operator** (Differential operator). Thus in this chapter we want to find solutions to the equation

\[ L(y) = g(x) \quad y(x_0) = y_0 \quad y'(x_0) = y'_0 \]
Linearity of the differential operator $L$:

Let $L(y) = y'' + p(x)y' + q(x)y$. If $y$, $y_1$, and $y_2$ are any twice-differentiable functions on the interval $I$ and if $c$ is any constant, then

1. $L(y_1 + y_2) = L(y_1) + L(y_2)$,
2. $L(cy) = cL(y)$

Proof:

(1)

(2) the proof of (2) is exercise #28 sec 4.2

Note: Any operator that satisfies (1) & (2) is called **linear operator**. If (1) & (2) fails to hold, the operator is **nonlinear**

(2) **D- Operator**

Recall from calculus that

$Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dy^2}$, and in general $D^n y = \frac{d^n y}{dy^n}$. Using this notations, we can express $L$ as

$L[y] = D^2 y + pDy + qy = (D^2 + pD + q)[y]$

for example, if

$L[y] = y'' + 4y' + 3y$

then we can write
\[ L[y] = D^2 y + 4Dy + 3y = (D^2 + 4D + 3)[y] \]

when \( p \) and \( q \) are constants we can treat \( D^2 y + pDy + qy \) as polynomial, so for the above example we can write

\[ L[y] = D^2 y + 4Dy + 3y = (D^2 + 4D + 3)[y] = (D + 1)(D + 3)y \]

Exercises:

# 3.1.1 Let \( L_2[y] = y'' - y \) compute

(a) \( L_2[\sin x] \)
(b) \( L_2[x^2] \)
(c) \( L_2[x^r] \)
(d) \( L_2[2e^{2x}] \)
Sec 3.2: Solution of Homogeneous Linear Second Order Differential Equations

Theory of Solutions

Here we will investigate solutions to homogeneous differential equations. Consider the homogeneous linear differential equation

\[(H)\quad a(x)y'' + b(x)y' + c(x)y = 0.\]

We have the following results:

(1) **General Solution for a Second-Order Homogeneous Linear Equation:**

**Theorem:** The 2\(^{nd}\) order homogeneous equation

\[a(x)y'' + b(x)y' + c(x)y = 0 \quad (1)\]

always has exactly two linearly independent solutions \(y_1(x)\) and \(y_2(x)\). The general solution to equation (1) is given by the formula

\[y(x) = c_1y_1 + c_2y_2\]

where the functions \(y_1, y_2\) is the pair of linearly independent solutions to equation (1).

The above theorem makes clear the importance of being able to decide whether or not a set of solutions \(y_1(x)\) and \(y_2(x)\) of

\[(H)\quad a(x)y'' + b(x)y' + c(x)y = 0.\]

Are linearly independent.

Methods to check if two functions are Linearly independent:

(1) **Linear Dependence and Independence using Definition:**

**Definition:** (Linear dependence)

Two functions \(y_2\) **and** \(y_2\) are said to be linearly independent on the interval \(I\), if there exist nonzero constants \(c_2\) **and** \(c_2\) such that for all \(x\) in \(I\)

\[c_1y_1 + c_2y_2 = 0 \quad \triangleright\]
If \( y_2 \) and \( y_2 \) are linearly independent then \( c_1 = c_2 = 0 \) is the only solution for
\[
c_1 y_1 + c_2 y_2 = 0
\]

Another way of stating this definition is, ratio test as follows:

Two functions \( y_2 \) and \( y_2 \) are linearly dependent in \( I \) if \( \frac{y_1}{y_2} = c \)

Two functions \( y_2 \) and \( y_2 \) are linearly independent in \( I \) if \( \frac{y_1}{y_2} \neq c \)

Remark: Proportionality of two functions is equivalent to their linear dependence.

(2) Linear Dependence and Independence using Wronskian:

It is not easy to use the above definition for higher order, instead we can use the wronskian.

The Wronskian: for two functions

Let \( y_1 \) and \( y_2 \) be two differentiable functions. \( y_1 \) and \( y_2 \) are linearly dependent (proportional) if and only if \( \frac{y_1}{y_2} = c \). An equivalent criteria for linearly dependent can be derived from the following:

\[
y_2
\]

- \( y_1 \) and \( y_2 \) are linearly dependent
- So \( \frac{y_1}{y_2} = c \)
- Differentiate both sides \( \left( \frac{y_1}{y_2} \right)' = 0 \)

\[
y_1 y_2' - y_1' y_2 = 0
\]

\( y_1 y_2' - y_1' y_2 \) is called the Wronskian of \( y_1 \) and \( y_2 \), that is

\[
\text{Wronskian } W(y_1, y_2) \text{ of } y_1 \text{ and } y_2 \text{ is}
\]

\[
W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.
\]
Therefore, we have the following:

| Two functions \( y_1 \) and \( y_2 \) are linearly dependent if \( W(y_1, y_2) = 0 \) |
| Two functions \( y_1 \) and \( y_2 \) are linearly independent if \( W(y_1, y_2) \neq 0 \) |

Now we have the following conclusion:

Let \( y_1 \) and \( y_2 \) be two differentiable functions. The Wronskian \( W(y_1, y_2) \), associated to \( y_1 \) and \( y_2 \), is the function

\[
W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = y_1(x)y'_2(x) - y'_1(x)y_2(x).
\]

We have the following important properties:

1. If \( y_1 \) and \( y_2 \) are two solutions of the equation \( y'' + p(x)y' + q(x)y = 0 \), then
   \[
   W(y_1, y_2)(x) \neq 0 \quad \text{for every} \quad x \quad \iff \quad \exists x_0 \text{ such that } W(y_1, y_2)(x_0) \neq 0.
   \]
   In this case, we say that \( y_1 \) and \( y_2 \) are linearly independent.

2. If \( y_1 \) and \( y_2 \) are two linearly independent solutions of the equation \( y'' + p(x)y' + q(x)y = 0 \), then any solution \( y \) is given by
   \[
   y = c_1 y_1 + c_2 y_2,
   \]
   for some constant \( c_1 \) and \( c_2 \). In this case, the set \( \{y_1, y_2\} \) is called the fundamental set of solutions.

Exercises:

# 3.1.3 Determine whether the given solutions of the following DE are linearly independent

a) \( y'' + 9y = 0 \) \( y_1 = \sin 3x \) \( y_2 = \cos 3x \)

b) \( y'' - y = 0 \) \( y_1 = e^x \) \( y_2 = 2e^x \)

c) \( y'' + y - 2y = 0 \) \( y_1 = e^x \) \( y_2 = 3e^{-2x} \)

There is a fascinating relationship between second order linear differential equations and the Wronskian. This relationship is stated below.
**Theorem: Abel's Theorem**

Let $y_1$ and $y_2$ be solutions on the differential equation

$$L(y) = y'' + p(x)y' + q(x)y = 0$$

where $p$ and $q$ are continuous on $[a,b]$. Then the Wronskian is given by

$$W(y_1, y_2)(x) = Ce^{-\int p(x)dx}$$

where $C$ is a constant depending on only $y_1$ and $y_2$, but not on $x$. The Wronskian is either zero for all $x$ in $[a,b]$ or always positive, or always negative.

**Proof**: 

...
Methods for solving second order Linear Homogeneous DE:

Section 3.3: Method (1) (For Constant coefficients)

Characteristic equation (Auxiliary equation)

A second order homogeneous equation with constant coefficients is written as

\[ ay'' + by' + cy = 0 \quad (a \neq 0) \]

where \( a, b \) and \( c \) are constant. This type of equation is very useful in many applied problems (physics, electrical engineering, etc.). Let us summarize the steps to follow in order to find the general solution:

if \( a = 0 \), we get the first order equation of the same family

\[ by' + cy = 0 \]

and this is first order separable differential equation which has a solution of the form

\[ y = Ae^{\lambda x}, \quad \lambda = -c/b \]

In the same way we can think that, \( y = e^{\lambda x} \) will be a solution of the second order equation

\[ ay'' + by' + cy = 0 \quad (a \neq 0) \]

so, by substituting \( y = e^{\lambda x} \) \( y' = \lambda e^{\lambda x} \), \( y = \lambda^2 e^{\lambda x} \), in the left-hand side of differential equation, we get

\[ a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0 \]

now, dividing both sides by \( y = e^{\lambda x} \), we obtain

the characteristic equation (Auxiliary equation)

\[ a\lambda^2 + b\lambda + c = 0 \]

This is a quadratic equation. Let \( \lambda_1 \) and \( \lambda_2 \) be its roots we have
\[ \lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

The actual form of the solution is strongly dependent on whether the roots are real versus complex or distinct versus repeated. In fact three special cases can be identified based on whether the term inside the radical is positive, negative, or zero. There are three cases as follows

**Three Special Cases**

**Case I. Real Distinct Roots:**

If \( \lambda_1 \) and \( \lambda_2 \) are distinct real numbers (this happens if \( b^2 - 4ac > 0 \)), then the general solution is

\[ y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \]

**Exercises 3.3.2** Find the general solution of \( y'' + 3y' - 4y = 0 \)

**Case II. Repeated Roots:**

If \( \lambda_1 = \lambda_2 \) (which happens if \( b^2 - 4ac = 0 \)), then the general solution is

\[ y = c_1 e^{\lambda_1 x} + c_2 xe^{\lambda_1 x} \]

**Exercises 3.3.4** Find the general solution of \( 4y'' - 4y' + y = 0 \)
Case III. Complex Conjugate Roots: (section 2.3)

If \( \lambda_1 \) and \( \lambda_2 \) are complex numbers (which happens if \( b^2 - 4ac < 0 \)),

Then \( \lambda_1, \lambda_2 = \frac{-b}{2a} \pm \frac{i\sqrt{4ac - b^2}}{2a} \)

And the general solution is

\[
y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x),
\]

where

\[
\alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}.
\]

Example: Find the general solution \( 2y'' + 2y' + 3y = 0 \)

Exercises 3.3.12 Solve \( y'' + 16y = 0 \) \( y(0) = 1 \) \( y'(0) = 0 \)
Justification of Case 3:

\[ y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \]

using Euler identity: \[ e^{\pm ix} = \cos x \pm i \sin x \]

the solution can be written as

\[ y = e^{\alpha x} [c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] \]

\[ = e^{\alpha x} ((c_1 + c_2) \cos \beta x + (c_1 - c_2)i \sin \beta x) \]

\[ = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) \quad \text{where} \quad C_1 = c_1 + c_2 \quad \text{and} \quad C_2 = (c_1 - c_2)i \]

Proof of Euler identity:

From Maclaurin series expansion

\[ e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots \]

\[ = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \ldots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]

From the above expansions, we conclude the following

\[ e^{\pm ix} = \cos x \pm i \sin x \quad \text{which is Euler identity} \]
Method (2): Euler Equations

An Euler-Cauchy equation is of the form

\[ ax^2 y'' + xy' + cy = 0 \quad x > 0 \]

where \( a, b \) and \( c \) are constant numbers. Let us consider the change of variable

\[ x = e^t. \]

Then we have

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} \quad \text{and} \quad \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} \cdot \frac{1}{x^2} + \frac{dy}{d^2 x} \cdot \frac{1}{x^2} = \frac{d^2 y}{dt^2} \cdot \frac{1}{x^2} + \frac{dy}{dt} \cdot \left(\frac{1}{x^2}\right)
\]

\[
= \frac{1}{x^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)
\]

Substitute in \((\text{EC})\)

The equation \((\text{EC})\) reduces to the new equation

\[
a \frac{d^2 y}{dt^2} + (b - a) \frac{dy}{dt} + cy = 0
\]

We recognize a second order differential equation with \textbf{constant coefficients}. Therefore, we use the previous sections to solve it. We summarize below all the cases:

1. Write down the characteristic equation
   \[ a \lambda^2 + (b - a) \lambda + c = 0 \]

2. If the roots \( r_1 \) and \( r_2 \) are distinct real numbers, then the general solution of \((\text{EC})\) is given by
   \[ y(x) = c_1 x^{r_1} + c_2 x^{r_2} \]

3. If the roots \( \lambda_1 \) and \( \lambda_2 \) are equal \( (\lambda_1 = \lambda_2) \), then the general solution of \((\text{EC})\) is
   \[ y(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_1} \ln x \]
If the roots $\lambda_1$ and $\lambda_2$ are complex numbers, then the general solution of (EC) is

$$y = c_1 x^\alpha \cos \beta \ln x + c_2 x^\alpha \sin \beta \ln x$$

**Short cut for using Euler method:**

We assume a solution of the form

$$y = x^\lambda$$

Therefore,

$$y' = \lambda x^{\lambda-1} \quad \text{and} \quad y'' = \lambda(\lambda - 1)x^{\lambda-2}$$

Upon substitution into the original ODE, we get

$$(a\lambda(\lambda - 1) + b\lambda + c)x^\lambda = 0$$

and dividing by $x^\lambda$ gives the characteristic equation,

$$a\lambda^2 + (b - a)\lambda + c = 0.$$
Example: Find the general solution to \( x^2 y''(x) + 2xy'(x) - 6y(x) = 0 \quad x>0 \)

Solution: First we recognize that the equation is an Euler equation, with \( a=1 \), \( b=2 \) and \( c=-6 \).

Exercise # 3.3.14 solve \( x^2 y'' + xy' + 4y = 0 \quad x>0 \)

Example: Find the general solution of

\[
(x + 1)^2 y''(x) + 10(x + 1)y'(x) + 14y(x) = 0 \quad x>-1, \quad \text{Ans: } c_1 (x + 1)^{-2} + c_2 (x + 1)^{-7}
\]
Method(3): For variable coefficients

\[ a(x)y'' + b(x)y' + c(x)y = 0. \]

Integration using one known solution ( Reduction of order method)

This technique is very important since it helps one to find a second solution independent from a known one. Therefore, according to the previous section, in order to find the general solution to \( y'' + p(x)y' + q(x)y = 0 \), we need only to find one (non-zero) solution, \( y_1 \).

Let \( y_1 \) be a non-zero solution of

\[ y'' + p(x)y' + q(x)y = 0. \]

Then, a second solution \( y_2 \) independent of \( y_1 \) can be found as

\[
y_2(x) = y_1(x) \int e^{-\int p(x)dx} \frac{1}{[y_1(x)]^2} dx
\]

Development of the Method

Given a valid solution to a homogeneous linear 2\(^{nd}\) order differential equation, one can obtain a second linearly independent solution by letting

\[ y_2(x) = u(x)y_1(x) \]

where \( y_1(x) \) is known and \( u(x) \) is to be determined. This technique applied to a 2\(^{nd}\) order system is as follows:

Given the original differential equation

\[ y''(x) + p(x)y'(x) + q(x)y(x) = 0 \quad (2.14) \]

with known solution \( y_1(x) \), we let

\[
y_2 = uy_1
\]

\[
y_2' = u'y_1 + uy_1'
\]
\[ y_2'' = u''y_1 + u'y_1' + uy_1' + uy_1'' \]

or

\[ y_2'' = u''y_1 + 2u'y_1' + uy_1'' \]

Now, substitution into the original equation gives

\[ u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0 \]

Or

\[ u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0 \]

Letting \( z = u' \) and noting that the third term vanishes from definition of the original ODE, we have

\[ z'y_1 + z(2y_1' + py_1) = 0 \quad (2.15) \]

Now, separating variables and integrating gives

\[ \frac{z'}{z} = \left( \frac{2y_1'}{y_1} + p \right) \]

\[ \ln z = -2\ln y_1 - \int p(x)dx \]

\[ \ln(zy_1^2) = -\int p(x)dx \]

Or

\[ z(x) = \frac{1}{y_1^2} e^{-\int p(x)dx} \quad (2.16) \]

Finally, since \( du/dx = z \), we have

\[ u(x) = \int z(x)dx \quad \text{and} \quad y_1(x) = u(x)y_1(x) \quad (2.17) \]

**Example** \( x^2 y'' + 6xy' + 6y = 0, \quad x > 0; \quad f(x) = x^{-2} \). Ans: \( y = x^{-3} \)
Sec 3.4 Nonhomogeneous Second Order Linear Equations

So far we have considered equation of the form

\[ ay'' + by' + cy = g(x) \]

for the case where \( g(x) = 0 \).

When \( g(x) = 0 \), then \( ar^2 + br + c = 0 \) giving \( r = r_1 \) and \( r = r_2 \) and the solution is in general

\[ y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \]

In the equation \( ay'' + by' + cy = g(x) \), the substitution \( y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} \) would make the left-hand side zero. Therefore, there must be a further term in the solution which will make the L.H.S equal to \( g(x) \) and not zero.

The complete solution will therefore be of the form

\[ y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + y_p \]

where \( y_p \) is the extra function yet to be found.

\[ y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} \]

is called the complementary solution or homogeneous solution.

\[ y = y_p(x) \]

is called the particular solution.

And \( \text{General solution} = \text{homogeneous solution} + \text{particular solution} \)

\[ y = y_p(x) + y_h(x) \]

In the previous sections we discussed how to find \( y_h \). In this section we will discuss two techniques to find \( y_p \).

1. Method of Principle of superposition:
2. Method of variation of parameters
Method (1): Principle of superposition:

Let $y$ be a solution of the differential equation

$$a y'' + b y' + cy = f_1(x)$$

and let $\hat{y}$ be a solution of

$$a \hat{y}'' + b \hat{y}' + c \hat{y} = f_2(x)$$

Then for any constants $C_1, C_2$, the function $C_1 y + C_2 \hat{y}$ is a solution of the differential equation

$$a y'' + b y' + cy = c_1 f_1(x) + c_2 f_2(x)$$

Example:

Given that $y_1(t) = \cos t$ is a solution to $y'' - y' + y = \sin t$

And $y_2(t) = e^{2t} / 3$ is a solution to $y'' - y' + y = e^{2t}$

Use the superposition principle to find solution to the following differential equations

(a) $y'' - y' + y = 5 \sin t$

(b) $y'' - y' + y = \sin t - 3e^{2t}$

(c) $y'' - y' + y = 4 \sin t + 18e^{2t}$
Method (2) : Variation of Parameters

This method has no prior conditions to be satisfied. Therefore, it may sound more general than the previous method. We will see that this method depends on integration while the previous one is purely algebraic which, for some at least, is an advantage. It is easily extended to higher order linear ODEs.

Consider the equation  
\[ y'' + p(x) y' + q(x) y = g(x) \]

In order to use the method of variation of parameters we need to know that \( \{y_1, y_2\} \) is a set of fundamental solutions of the associated homogeneous equation \( y'' + p(x) y' + q(x) y = 0 \). We know that, in this case, the general solution of the associated homogeneous equation is \( y_h = c_1 y_1 + c_2 y_2 \). The idea behind the method of variation of parameters is to look for a particular solution such as

\[ y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x), \]

where \( u_1 \) and \( u_2 \) are functions. From this, the method got its name. The functions \( u_1 \) and \( u_2 \) can be calculated as follows,

Let’s start with the general 2\(^{nd}\) order equation written in standard form,

\[ y'' + p(x) y' + y = g(x) \]

with homogeneous solution

\[ y_h(x) = c_1 y_1 + c_2 y_2 \]

Now assume \( y_p \) of the form

\[ y_p(x) = u_1(x) y_1 + u_2(x) y_2 \]  \hspace{1cm} (1)

which gives,

\[ y'_p = u_1 y'_1 + u'_1 y_1 + u_2 y'_2 + u'_2 y_2 \]

At this point we recognize that an addition constraint, beyond the original balance equation (1), will be needed because we are trying to find two unknowns, \( u_1 \) and \( u_2 \). For simplicity, let’s choose the following relationship,

\[ u'_1 y_1 + u'_2 y_2 = 0 \]  \hspace{1cm} (2)

which greatly simplifies the above equation for \( y'_p \), giving

\[ y'_p = u_1 y'_1 + u_2 y'_2 \]
Now the second derivative becomes
\[ y''_p = u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2 \]

Substitution of these expressions into the original differential equation gives
\[ u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2 + pu_1 y'_1 + pu_2 y'_2 + qu_1 y_1 + qu_2 y_2 = g(x) \]

Rearranging gives
\[ u_1 (y''_1 + py'_1 + qy_1) + u_2 (y''_2 + py'_2 + qy_2) + u'_1 y'_1 + u'_2 y'_2 = g(x) \]

Therefore, since both the first and second terms vanish (because \( y_1 \) and \( y_2 \) are solutions to the homogeneous equation), we have
\[ u'_1 y'_1 + u'_2 y'_2 = g(x) \quad (3) \]

The functions \( u_1 \) and \( u_2 \) are solutions to the system of equations (2) and (3)
\[ \begin{cases} u'_1 y_1 + u'_2 y_2 = 0 \\ u'_1 y'_1 + u'_2 y'_2 = g(x) \end{cases} \]

which can be written in matrix form (two equations with two unknowns) gives
\[
\begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}
\]

Using Cramer’s rule the solution to this system can be written as
\[
u'_1 = \begin{vmatrix} 0 & y_2 \\ g & y'_2 \end{vmatrix} = \begin{vmatrix} 0 & y_2 \\ g & y'_2 \end{vmatrix} g \\
y_1 & y_2 \\
y'_1 & y'_2 \\
\]

and
\[
u'_1 = \begin{vmatrix} y_1 & 0 \\ y'_1 & g \end{vmatrix} = \begin{vmatrix} y_1 & 0 \\ y'_1 & g \end{vmatrix} g \\
y_1 & 0 \\
y'_1 & 1 \\
\]

Finally, integrating these latter expressions to give \( u_1 \) and \( u_2 \) allows one to write the desired expression for \( y_p(x) \) as
\[
\begin{align*}
\psi_1(x) &= -\int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)} \, dx, \\
\psi_2(x) &= \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)} \, dx.
\end{align*}
\]

This result is consistent with the general relationships given above.

\[
y_p(x) = -y_1(x) \int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)} \, dx + y_2(x) \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)} \, dx.
\]

Exercise 3.4.6 find the general solution of \( y'' + y = \sec x \).
Exercise # 3.4.10 Solve using variation of parameters

\[ x^2 y'' + 5xy' + 3y = x^{-1} \]